

HIGH-DIMENSIONAL FILLINGS IN HEISENBERG GROUPS

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ABSTRACT. We use intersections with horizontal manifolds to show that high-dimensional cycles in the Heisenberg group can be approximated efficiently by simplicial cycles. This lets us calculate all of the higher-order Dehn functions of the Heisenberg groups. By applying a similar technique to general nilpotent Lie groups with lattices, we recover a Sobolev inequality of Varopoulos.

1. INTRODUCTION

Nilpotent groups, especially Carnot groups, have scaling properties that make them particularly useful in geometry. They appear, for instance, as tangent cones of sub-riemannian manifolds and in the horospheres of negatively-curved symmetric spaces. Their scaling automorphisms have also been used to solve problems on fillings and extensions of Lipschitz maps to Carnot groups. Gromov, for instance, used scaling techniques and microflexibility to construct extensions from Lipschitz maps of spheres to Lipschitz maps of discs and bound the Dehn functions of some nilpotent groups [Gro96]. In [You08], we found techniques for avoiding the use of microflexibility and generalizing Gromov's Dehn function techniques to higher-order Dehn functions, and Stefan Wenger and the author used these techniques to give an elementary proof of Gromov's Lipschitz extension theorem [WY09].

One way of measuring the difficulty of extension problems is the study of filling invariants such as the higher-order Dehn functions, FV^d . In [Gro93], Gromov made the following conjecture:

Conjecture 1. *In the $(2n+1)$ -dimensional Heisenberg group H_{2n+1} ,*

$$\begin{aligned} FV^{d+1}(V) &\sim V^{\frac{d+1}{d}} && \text{for } 1 \leq d < n \\ FV^{d+1}(V) &\sim V^{\frac{d+2}{d}} && \text{for } d = n, \\ FV^{d+1}(V) &\sim V^{\frac{d+2}{d+1}} && \text{for } n < d < 2n+1. \end{aligned}$$

In [You08], we proved the first two bounds; in this paper, we will prove the third.

We will also use the same techniques to prove a bound on the top-dimensional filling invariants.

Theorem 2. *If G is a Carnot group of dimension n which supports a lattice and κ is the volume growth exponent of G , then*

$$FV_G^n(V) \lesssim V^{\frac{\kappa}{\kappa-1}}.$$

The corresponding lower bound is achieved by spheres in G of appropriate radii. The theorem also follows from the Sobolev inequalities for nilpotent groups proved by Varopoulos [Var88]; indeed, the theorem gives an alternate proof of certain Sobolev inequalities for nilpotent groups. Mukherjee [Muk08] proved Thm. 2 in the case that G is the 3-dimensional Heisenberg group.

The main technique we used to construct these fillings and extensions in [You08] was approximation by horizontal maps. Recall that Carnot groups have a family of scaling automorphisms and that this family of automorphisms stretches different directions in the nilpotent group by different amounts. The directions which are scaled the least are called *horizontal directions*, and maps tangent to these directions are called *horizontal maps*. Some Carnot groups have an abundance of horizontal maps. Maps to these groups can be approximated by horizontal maps by constructing a triangulation of the group which has horizontal faces and using the Federer-Fleming Deformation Theorem or a similar simplicial approximation technique to approximate maps from k -manifolds by maps to the k -skeleton of the triangulation.

We used the scaling automorphism to vary the size of the simplices in the triangulation and thus the characteristics of the approximation. If the simplices are small, a map can be approximated very closely, but the approximation has many simplices. If the simplices are large, the approximation is simpler but less accurate.

To fill a map f , we first approximated it by maps f_i where f_i approximates f in a skeleton with simplices of diameter $\sim 2^i$. We then connected approximations of different scales, finding homotopies between f_i and f_{i+1} for all i . This gives a homotopy from f to f_i for all i , and if i is large enough, it gives a filling of f . The volume of the filling constructed this way depends on the sizes of the f_i , so the problem of finding efficient fillings of cycles in a nilpotent group becomes the problem of finding efficient approximations.

In [You08], we used techniques based on Federer and Fleming's Deformation Theorem [FF60] to construct these approximations; if f is a d -cycle of mass V in G and τ is a triangulation of G with bounded geometry (e.g. a triangulation which is equivariant under the action of a cocompact lattice in G), then the Deformation Theorem implies that f can be approximated by a simplicial d -cycle which is a sum of roughly V simplices of τ . We call this approximation f_0 . To obtain the rest of the f_i , we used scaling automorphisms. If $s_t : G \rightarrow G, t > 0$ is the family of scaling automorphisms, then $s_{2^{-i}} \circ f$ has mass $\leq 2^{-di}V$. Approximating it in τ gives a cycle of mass $\lesssim 2^{-di}V$, and scaling this by s_{2^i} gives a sum of $\lesssim 2^{-di}V$ cells in $s_{2^i}(\tau)$ which approximates f ; this is f_i .

This technique sufficed for the applications in [You08], but does not work as well for the general case. In that paper, we could ensure that d -cells of τ were horizontal, in which case a d -cell of $s_{2^i}(\tau)$ has volume $\sim 2^{di}$, and the f_i constructed above has volume $\sim V$. In general, however, there is some $k = k(d) \geq d$ such that d -cells of $s_t(\tau)$ have volume $\sim t^k$, so f_i has volume $\sim V2^{(k-d)i}$, which may be much larger than the optimum. In this paper, we will give better methods for finding approximations in these settings and use them to prove filling inequalities.

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2. PRELIMINARIES

There are different definitions of higher-order Dehn functions, with similarities and differences. We will mainly work with a definition based on Lipschitz chains, as in [Gro83]. Descriptions of other versions of higher-order filling invariants can be found in [AWP99, ECH⁺92, Gro09a, Gro09b]. Recall that an integral Lipschitz d -chain in a space X is a finite linear combination, with integer coefficients, of

Lipschitz maps from the euclidean d -dimensional simplex Δ^d to X . We will often call this simply a Lipschitz d -chain.

We will generally take X to be a riemannian manifold, so a Lipschitz map to X is differentiable almost everywhere, and one can define the volume of a Lipschitz map as the integral of the magnitude of its jacobian. If a is a Lipschitz d -chain and $a = \sum_i x_i \alpha_i$ for some maps $\alpha_i : \Delta^d \rightarrow X$ and some coefficients $x_i \in \mathbb{Z}, x_i \neq 0$, we define

$$\|a\|_1 := \sum_i |x_i|.$$

When $d > 0$, we define the *mass* of a to be

$$\text{mass } a := \sum_i x_i \text{vol}_d \alpha_i,$$

and if $d = 0$, we let $\text{mass } a := \|a\|_1$. We define the *support* of a to be

$$\text{supp } a = \bigcup_i \alpha_i(\Delta^d).$$

The Lipschitz chains form a chain complex, which we denote $C_*^{\text{lip}}(X)$. If $g : X \rightarrow Y$ is a Lipschitz map, we can define $g_\#$, the pushforward map, to be the linear map which sends the simplex $\alpha : \Delta^d \rightarrow X$ to the simplex $g \circ \alpha$; this is a map of chain complexes. Likewise, we can define a smooth chain to be a sum of smooth maps of simplices.

Our approximations are based on triangulations. A *triangulation* of a space X consists of a simplicial complex τ and a homeomorphism $\phi : \tau \rightarrow X$. We will generally refer to the tuple (τ, ϕ) as τ , leaving the homeomorphism implicit. We can put a metric and a smooth structure on τ so that each simplex is isometric to the standard euclidean simplex; we say that a map $\tau \rightarrow X$ is *piecewise smooth* if it is smooth on each simplex. We will require throughout this paper that ϕ is locally Lipschitz.

If τ is a triangulation of X whose faces are smooth or Lipschitz, then simplicial chains of X are formal sums of faces, and we will consider the simplicial chains of X as a subcomplex of the smooth or Lipschitz chains. We denote the chain complex of simplicial chains of τ by $C_*(\tau)$. Federer and Fleming showed that Lipschitz chains can be approximated by simplicial chains [FF60]; we will only require approximations of Lipschitz cycles, so we state a simpler version of the theorem. In this theorem, we think of $P_\tau^{\text{ff}}(\alpha)$ as an approximation of α and of $Q_\tau^{\text{ff}}(\alpha)$ as a chain which interpolates between α and $P_\tau^{\text{ff}}(\alpha)$.

Theorem 3 (Deformation Theorem [FF60, ECH⁺92]). *Let τ be a triangulation of X such that each simplex of τ is bilipschitz equivalent to the standard euclidean simplex and assume that the simplices of τ fall into finitely many isometry classes.*

There is a constant c depending on τ such that if $\alpha \in C_k^{\text{lip}}(X)$ is a cycle, then there are $P_\tau^{\text{ff}}(\alpha) \in C_k(\tau)$ and $Q_\tau^{\text{ff}}(\alpha) \in C_{k+1}^{\text{lip}}(X)$ such that

- (1) $\text{mass } P_\tau^{\text{ff}}(\alpha) \leq c \cdot \text{mass}(\alpha)$;
- (2) $\|Q_\tau^{\text{ff}}(\alpha)\| \leq c \cdot \text{mass}(\alpha)$; and
- (3) $\partial Q_\tau^{\text{ff}}(\alpha) = \alpha - P_\tau^{\text{ff}}(\alpha)$.

Furthermore, $P_\tau^{\text{ff}}(\alpha)$ and $Q_\tau^{\text{ff}}(\alpha)$ are supported in the smallest subcomplex of τ which contains the support of α .

Federer and Fleming originally proved their theorem in the case of Lipschitz currents in \mathbb{R}^n , so this statement is somewhat different from their original; it is closest to the version proved in [ECH⁺92].

If X is a d -connected riemannian manifold and a is an integral Lipschitz d -cycle in X , we define the filling volume of a to be:

$$\mathrm{FV}_X^{d+1}(a) := \inf_{\partial b = a} \mathrm{mass} b,$$

where the infimum is taken over the set of $b \in C_{d+1}^{\mathrm{lip}}(X)$ such that $\partial b = a$. We get the $(d+1)$ -dimensional filling volume function by taking a supremum over all cycles of a given volume:

$$\mathrm{FV}_X^{d+1}(V) := \sup_{\mathrm{mass} a \leq V} \mathrm{FV}_X^{d+1}(a),$$

where the supremum is taken over integral Lipschitz d -cycles.

A related definition is the d -th order Dehn function, δ^d , which measures the volume necessary to extend a map $S^d \rightarrow X$ to a map $D^{d+1} \rightarrow X$ (the fact that δ^d corresponds most closely to FV^{d+1} is unfortunate but conventional). If X is a d -connected manifold or simplicial complex and $f : S^d \rightarrow X$ is a Lipschitz map, we define

$$\delta_X^d(f) := \inf_{\substack{g : D^{d+1} \rightarrow X \\ g|_{S^d} = f}} \mathrm{vol}_d g$$

and

$$\delta_X^d(V) := \sup_{\substack{f : S^d \rightarrow X \\ \mathrm{vol}_d f \leq V}} \delta_X^d(f).$$

We have defined δ^d and FV^{d+1} as invariants of spaces, but they can in fact be defined as invariants of groups. If G is a group which acts geometrically (that is, co-compactly, properly discontinuously, and by isometries) on a d -connected manifold or simplicial complex X , then the asymptotic growth rates of $\delta_X^d(V)$ and $\mathrm{FV}_X^{d+1}(V)$ are invariants of G . To make this rigorous, we can define a partial ordering on functions $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ so that $f \lesssim g$ if and only if there is a c such that

$$f(x) \leq cg(cx + c) + cx + c.$$

We say $f \sim g$ if and only if $f \lesssim g$ and $g \lesssim f$. Then if X_1 and X_2 are two d -connected manifolds or simplicial complexes on which G acts geometrically, then

$$\delta_{X_1}^d(V) \sim \delta_{X_2}^d(V)$$

and

$$\mathrm{FV}_{X_1}^d(V) \sim \mathrm{FV}_{X_2}^d(V).$$

This is proved for a simplicial version of δ^d in [AWP99], but the proof there also applies to a simplicial version of FV^d ; one can show that the Lipschitz versions used here are equivalent to simplicial versions using the Deformation Theorem.

The relationship between δ^d and FV^{d+1} depends on d . When $d \geq 3$, then $\delta_X^d \sim \mathrm{FV}_X^{d+1}$ for all d -connected manifolds or simplicial complexes X . When $d = 2$, then $\delta_X^d \lesssim \mathrm{FV}_X^{d+1}$. (See [Gro83, App. 2.(A')] for the upper bound on δ^d and [BBFS09, Rem. 2.6.(4)] for the lower bound; see also [Gro09a, Gro09b].) Since this paper focuses on upper bounds on FV^{d+1} when $d \geq 2$, the bounds in this paper will also hold for δ^d .

In this paper, we focus on finding filling inequalities for Carnot groups, which are nilpotent Lie groups provided with a family of scaling automorphisms. Recall that if G is a simply-connected nilpotent Lie group and \mathfrak{g} is its Lie algebra, then the lower central series

$$\mathfrak{g} = \mathfrak{g}_0 \supset \cdots \supset \mathfrak{g}_{k-1} = \{0\},$$

$$\mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}]$$

terminates. If $\mathfrak{g}_k = \{0\}$ and $\mathfrak{g}_{k-1} \neq \{0\}$, we say that \mathfrak{g} has nilpotency class k . If there is a decomposition

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k$$

such that

$$\mathfrak{g}_i = V_{i+1} \oplus \cdots \oplus V_k$$

and $[V_i, V_j] \subset V_{i+j}$ for all $i, j \leq k$, we call it a grading of \mathfrak{g} . If \mathfrak{g} has a grading, we can extend the V_i to left-invariant distributions on G and give G a left-invariant metric such that the V_i 's are orthogonal. With this metric, G is called a Carnot group.

If G is a Carnot group, there is a family of automorphisms $s_t : G \rightarrow G$ which act on the Lie algebra by $s_t(v) = t^i v$ for all $v \in V_i$. These automorphisms distort vectors in \mathfrak{g} by differing amounts. Vectors in V_1 are distorted the least, and we call these vectors *horizontal*. If M is a manifold and $f : M \rightarrow G$ is a piecewise smooth map, we say that f is *horizontal* if the tangent planes to f all lie in the distribution V_1 .

3. DUAL SKELETA AND APPROXIMATIONS

As described in the introduction, we will construct fillings of chains in a nilpotent group by approximating those chains at different scales. In this section, we will give a method for constructing these approximations using intersections with dual skeleta. This is roughly similar to the Deformation Theorem; given a d -cycle α , we will construct an approximation $P_\tau(\alpha)$ and a $(d+1)$ -chain $Q_\tau(\alpha)$ such that $\partial Q_\tau(\alpha) = \alpha - P_\tau(\alpha)$.

Let τ be a triangulation of an oriented riemannian manifold X of dimension n . Let $B(\tau)$ be the barycentric subdivision of τ . If σ is a d -simplex of τ , we can construct a subcomplex $\sigma^* \subset B(\tau)$ which is a disc which intersects σ in a point. If σ' is a simplex of τ , let $p_{\sigma'}$ be the barycenter of σ' ; this is a vertex of $B(\tau)$. Let

$$S = \{p_{\sigma'} | \sigma' \in \tau, \sigma \subset \sigma'\},$$

and let σ^* be the subcomplex of $B(\tau)$ consisting of all simplices with vertices in S . Since X is a manifold, σ^* is homeomorphic to a disc of dimension $n-d$. The interiors of the σ^* partition X into disjoint sets; in fact, they give X the structure of a CW-complex, which we call the *dual complex* and denote by τ^* .

If $f : \tau_1 \rightarrow X$ and $g : \tau_2 \rightarrow X$ are piecewise smooth maps from two simplicial complexes, we say that they are *transverse* if for all simplices $\Delta_1 \in \tau_1$ and $\Delta_2 \in \tau_2$, the restrictions $\alpha|_{\text{int } \Delta_1}$ and $\beta|_{\text{int } \Delta_2}$ are transverse. If a and b are smooth chains, we say they are transverse if $a = \sum x_i \alpha_i$ and $b = \sum y_j \beta_j$ for some maps α_i and β_j such that α_i is transverse to β_j for all i and j .

If X is an oriented manifold and $\alpha : \Delta^d \rightarrow X$ and $\beta : \Delta^{d'} \rightarrow X$ are transverse, where Δ^d and $\Delta^{d'}$ are the standard simplices of dimension d and d' and $d+d' = n$, we can define $i(\alpha, \beta)$ to be the intersection number of α and β . We can extend i linearly to define $i(a, b)$ when a and b are transverse smooth chains. Since the

intersection number is invariant under small perturbations, we can also define $i(a, b)$ for some non-transverse, non-smooth chains; it suffices that a and b are Lipschitz chains such that $\text{supp } \partial a \cap \text{supp } b = \emptyset$ and $\text{supp } a \cap \text{supp } \partial b = \emptyset$.

Remark. This intersection number is defined for generic a and b . For instance, if $X = G$ is a nilpotent Lie group and a and b are Lipschitz chains of complementary dimension, then $\text{supp } \partial(g \cdot a) \cap \text{supp } b = \emptyset$ and $\text{supp } g \cdot a \cap \text{supp } \partial b = \emptyset$ for all but a measure zero set of $g \in G$, so $i(g \cdot a, b)$ is defined for all but a measure zero set of $g \in G$.

Let τ be a triangulation of X , with corresponding homeomorphism $\phi : \tau \rightarrow X$. For every simplex σ of τ , let $\hat{\sigma}$ and $\widehat{\sigma^*}$ be the fundamental classes of σ and σ^* . Then $\phi_{\#}(\hat{\sigma})$ and $\phi_{\#}(\widehat{\sigma^*})$ are Lipschitz chains in X and we can require that they be oriented so that $i(\phi_{\#}(\hat{\sigma}), \phi_{\#}(\widehat{\sigma^*})) = 1$. By abuse of notation, we will identify chains and cycles in τ with their images in X when it is clear, and simply write $i(\hat{\sigma}, \widehat{\sigma^*})$.

Let a be a Lipschitz d -chain in X . If $i(a, \widehat{\sigma^*})$ is defined for all cells σ in $\tau^{(d)}$, define

$$P_{\tau}(a) := \sum_{\sigma \in \tau^{(d)}} i(a, \widehat{\sigma^*}) \sigma \in C_{d+1}(\tau),$$

where $\tau^{(d)}$ is the d -skeleton of τ , considered as the set of d -cells. This is a simplicial d -chain which approximates a , and standard arguments show that

$$\partial P_{\tau}(a) = P_{\tau}(\partial a).$$

Furthermore, if a is a simplicial chain, then $P_{\tau}(a) = a$.

It will be useful to have a generalization of this construction. Let τ be a triangulation of X and let $f : \tau \rightarrow X$ be a piecewise smooth map; note that f need not be a homeomorphism. If $i(a, f_{\#}(\widehat{\sigma^*}))$ is defined for all cells σ in τ , define

$$P_{f(\tau)}(a) := \sum_{\sigma \in \tau^{(d)}} i(a, f_{\#}(\widehat{\sigma^*})) f_{\#}(\sigma) \in C_d^{\text{sm}}(X).$$

The key to proving bounds on filling functions is to find bounds on mass $P_{\tau}(a)$ and $\|P_{\tau}(a)\|_1$. If a is a chain, it might intersect the cells of τ^* in complicated ways, and mass $P_{\tau}(a)$ could be much larger than mass a , but we will show that in a nilpotent Lie group, this situation is unusual and can be avoided by translating a or τ ; this generalizes an analogous result of Federer and Fleming for \mathbb{R}^n [FF60]. The following lemma will be helpful:

Lemma 4. *Let G be a nilpotent Lie group. There is a $c_{\cap} > 0$ depending on G such that if $m + n = \dim G$, a is an m -dimensional smooth chain, and b is an n -dimensional smooth chain, then*

$$\int_G |i(g \cdot a, b)| dg \leq c_{\cap} \text{mass } a \text{ mass } b.$$

Proof. We can consider the case that a and b both consist of single smooth simplices of diameter ≤ 1 . Any other chain can be barycentrically subdivided until all of its simplices have diameter ≤ 1 , so the general case follows by linearity.

First, note that because G is unimodular, for all $h \in G$, we have

$$\begin{aligned} \int_G |i(g \cdot a, b)| dg &= \int_G |i(h^{-1}gh \cdot a, b)| dg \\ &= \int_G |i(gh \cdot a, h \cdot b)| dg, &= \int_G |i(g'h' \cdot a, h \cdot b)| dg', \end{aligned}$$

where $g' = gh h^{-1}$, so we may replace a and b with $h' \cdot a$ and $h \cdot b$ respectively. We may thus assume that a and b are contained in a ball of radius 1 around $0 \in G$.

Let Δ^m and Δ^n be the standard euclidean simplices of dimension m and n , and let $\alpha : \Delta^m \rightarrow G$ and $\beta : \Delta^n \rightarrow G$ be the maps corresponding to a and b respectively. Let $\gamma : \Delta^m \times \Delta^n \rightarrow G$ be the map $(x, y) \mapsto \beta(y)\alpha(x)^{-1}$. For all but a measure zero set of $g \in G$, the maps $g \cdot \alpha$ and β are transverse. Each of their intersections then corresponds to an x and y such that $g\alpha(x) = \beta(y)$ and thus an element of $\gamma^{-1}(g)$. In particular, $|i(g \cdot a, b)| \leq \#\gamma^{-1}(g)$. Since $m + n = \dim G$, we have

$$\int_G \#\gamma^{-1}(g) dg \leq \text{vol } \gamma,$$

it suffices to bound $\text{vol } \gamma$. We can write γ as the composition of maps $\gamma_0 : \Delta^m \times \Delta^n \rightarrow G \times G$ and $p : G \times G \rightarrow G$, where $\gamma_0(x, y) = (\alpha(x), \beta(y))$ and $p(h, k) = kh^{-1}$. The image of γ_0 is contained in a ball of radius 2; if we denote this ball by B , we have

$$\text{vol } \gamma \leq (\text{vol } \gamma_0)(\text{Lip } p|_B)^{\dim G} = (\text{vol } \alpha)(\text{vol } \beta)(\text{Lip } p|_B)^{\dim G},$$

where $\text{Lip } p|_B$ is the Lipschitz constant of $p|_B$. Thus the lemma holds for $c_\cap := (\text{Lip } p|_B)^{\dim G}$. \square

In the rest of this section, G will be a 1-connected nilpotent Lie group, and Γ will be a lattice in G . Let $(\tau, \phi : \tau \rightarrow G)$ be a triangulation of G , and let $\rho_x : G \rightarrow G$ be the map $\rho_x(g) = xg$. The group G acts on the set of triangulations; if $x \in G$, we let $x \cdot \tau$ be the triangulation $(\tau, \rho_x \cdot \phi)$.

We say that τ is Γ -adapted if it descends to a triangulation of the quotient $\Gamma \backslash G$. In this case, the lattice Γ acts on τ , the map ϕ is equivariant with respect to this action, and the simplices of τ fall into finitely many Γ -orbits. If τ is Γ -adapted and σ is a simplex of τ , we will write $\Gamma \cdot \sigma$ to denote the corresponding Γ -orbit of simplices. We will write $\Gamma \backslash \tau$ to denote the set of orbits of simplices.

It will be useful to consider approximations not only in τ , but also in translations of τ . Note that if $P_\tau(x^{-1} \cdot a)$ is defined, then

$$P_{x \cdot \tau}(a) = x \cdot P_\tau(x^{-1} \cdot a).$$

With a mild abuse of notation, we can let $P_{x \cdot f(\tau)}(a) := P_{(\rho_x \circ f)(\tau)}$.

Lemma 5. *Let τ be a Γ -adapted triangulation of G and let $f : \tau \rightarrow G$ be a Γ -equivariant piecewise smooth map. Let $Z \subset G$ be a compact fundamental domain for the right action of Γ on G , so that $Z\Gamma = G$. Let $d \in \mathbb{Z}$. If c_\cap is as in Lemma 4, then for all smooth d -cycles α ,*

$$\int_Z \|P_{x \cdot f(\tau)}(\alpha)\|_1 dx \leq c_\cap \text{mass } \alpha \sum_{\Gamma \cdot \sigma \in \Gamma \backslash \tau^{(d)}} \text{mass } f_\#(\widehat{\sigma^*}).$$

Consequently, there is an $x \in G$ such that

$$\|P_{x \cdot f(\tau)}(\alpha)\|_1 \leq c_\cap \frac{\text{mass } \alpha}{\text{vol } Z} \sum_{\Gamma \cdot \sigma \in \Gamma \backslash \tau^{(d)}} \text{mass } f_\#(\widehat{\sigma^*}).$$

Proof. Using Lemma 4, we see that

$$\begin{aligned} \int_Z \|P_{x \cdot f(\tau)}(\alpha)\|_1 dx &= \int_Z \sum_{\sigma \in \tau^{(d)}} |i(\alpha, x \cdot f_{\sharp}(\widehat{\sigma^*}))| dx \\ &\leq \sum_{\Gamma \cdot \sigma \in \Gamma \backslash \tau^{(d)}} \int_G |i(\alpha, x \cdot f_{\sharp}(\widehat{\sigma^*}))| dx \\ &\leq c_{\cap} \text{mass } \alpha \sum_{\Gamma \cdot \sigma \in \Gamma \backslash \tau^{(d)}} \text{mass } f_{\sharp}(\widehat{\sigma^*}), \end{aligned}$$

as desired. \square

If τ_1 and τ_2 are different triangulations of X , we can construct a chain interpolating between $P_{\tau_1}(\alpha)$ and $P_{\tau_2}(\alpha)$. Let τ be a triangulation of $X \times [0, 1]$ which restricts to τ_1 and τ_2 on $X \times \{0\}$ and $X \times \{1\}$ respectively. Consider the $(d+1)$ -chain

$$(1) \quad H = P_{\tau}(\alpha \times [0, 1]).$$

Its boundary is

$$\partial H = P_{\tau}(\alpha \times \{0\} - \alpha \times \{1\}) = P_{\tau_1}(\alpha) \times \{0\} - P_{\tau_2}(\alpha) \times \{1\},$$

so the projection of H to X has boundary $P_{\tau_1}(\alpha) - P_{\tau_2}(\alpha)$. To bound the volume of such constructions, we need a corollary of Lemma 4:

Corollary 6. *If $m+n = \dim G$, a is an m -dimensional smooth chain in G , and b is an n -dimensional smooth chain in $G \times [0, 1]$, then $i(g \cdot a \times [0, 1], b)$ is defined for all $g \in G$ except for a measure 0 subset, and*

$$\int_G |i(g \cdot a \times [0, 1], b)| dg \leq c_{\cap} \text{mass } a \text{mass } b.$$

Proof. As before, we can reduce to the case that a and b are single smooth simplices, given by maps α and β . Let $p : G \times [0, 1] \rightarrow G$ be the projection map and consider $p \circ \beta$. Each transverse intersection between $g \cdot \alpha$ and $p \circ \beta$ corresponds to a transverse intersection between $g \cdot \alpha \times [0, 1]$ and β , and vice versa, so

$$\int_G |i(g \cdot \alpha \times [0, 1], \beta)| dg = \int_G |i(g \cdot \alpha, p \circ \beta)| dg \leq c_{\cap} \text{vol } \alpha \text{vol } \beta$$

\square

If τ is a triangulation of $G \times [0, 1]$, we say that it is Γ -adapted if it descends to a triangulation of $\Gamma \backslash G \times [0, 1]$. We can then bound the size of H in (1):

Lemma 7. *Let τ be a Γ -adapted smooth triangulation of $G \times [0, 1]$, and let $f : \tau \rightarrow G \times [0, 1]$ be a Γ -equivariant map which is piecewise smooth. Let $d \in \mathbb{Z}$ and let Z be a compact fundamental domain for the right action of Γ on G . For all smooth d -cycles α in G ,*

$$\int_Z \|P_{x \cdot f(\tau)}(\alpha \times [0, 1])\|_1 dx \leq c_{\cap} \text{mass } \alpha \sum_{\Gamma \cdot \sigma \in \Gamma \backslash \tau^{(d+1)}} \text{mass } f_{\sharp}(\widehat{\sigma^*}).$$

Consequently, there is an $x \in G$ such that

$$\|P_{x \cdot f(\tau)}(\alpha \times [0, 1])\|_1 \leq c_{\cap} \frac{\text{mass } \alpha}{\text{vol}_n Z} \sum_{\Gamma \cdot \sigma \in \Gamma \backslash \tau^{(d+1)}} \text{mass } f_{\sharp}(\widehat{\sigma^*}).$$

Proof. Corollary 6 implies that

$$\begin{aligned} \int_Z \|P_{x \cdot f(\tau)}(\alpha \times [0, 1])\|_1 dx &\leq \sum_{\Gamma \cdot \sigma \in \Gamma \backslash \tau^{(d+1)}} \int_G |i(\alpha \times [0, 1], x \cdot f_{\#}(\widehat{\sigma^*}))| dx \\ &\leq c_{\cap} \text{mass } \alpha \sum_{\Gamma \cdot \sigma \in \Gamma \backslash \tau^{(d+1)}} \text{mass } f_{\#}(\widehat{\sigma^*}), \end{aligned}$$

as desired. \square

Finally, since α is generally close to $P_{f(\tau)}(\alpha)$, a standard chain homotopy argument often lets us construct a chain of small volume which interpolates between α and $P_{f(\tau)}(\alpha)$:

Lemma 8. *Let τ be a Γ -adapted triangulation of G and let $f : \tau \rightarrow G$ be a Γ -equivariant piecewise smooth map. For every smooth d -cycle α which is transverse to f , there is a smooth $(d+1)$ -chain $Q_{f(\tau)}(\alpha)$ such that $\partial Q_{f(\tau)}(\alpha) = \alpha - P_{f(\tau)}(\alpha)$.*

Furthermore, if Z is a fundamental domain for the right action of Γ , then there is a c_Q independent of α such that for all simplicial d -cycles α ,

$$\int_Z \text{mass } Q_{f(\tau)}(x^{-1} \cdot \alpha) dx \leq c_Q \text{mass } \alpha.$$

Proof. Since G is a 1-connected nilpotent Lie group, it is contractible. One contraction is given by the scaling automorphisms; let $h : G \times [0, 1] \rightarrow G$ be the map $h(g, t) = s_t(g)$. This is smooth, and if γ is a smooth k -cycle in G , then $h_{\#}(\gamma \times [0, 1])$ is a smooth $(k+1)$ -chain which fills γ . Furthermore, there is a $w : \mathbb{R} \rightarrow \mathbb{R}$ such that if γ is supported in the ball of radius r around 0, then $h_{\#}(\gamma \times [0, 1])$ is supported in the same ball and

$$\text{mass } h_{\#}(\gamma \times [0, 1]) \leq w(r) \text{mass } \gamma.$$

If γ is a non-zero smooth cycle in G , let $g_{\gamma} \in G$ be a point in the support of γ . Define $h' : C_*^{\text{lip}}(G) \rightarrow C_{*+1}^{\text{lip}}(G)$ by letting $h'(0) = 0$ and

$$h'(\gamma) = g_{\gamma} \cdot h_{\#}(g_{\gamma}^{-1} \gamma \times [0, 1])$$

for all $\gamma \neq 0$. If γ is a cycle and $\dim \gamma \geq 1$, then $\partial h'(\gamma) = \gamma$ and

$$\text{mass } h'(\gamma) \leq w(\text{diam } \gamma) \text{mass } \gamma,$$

where $\text{diam } \gamma$ is the diameter of the support of γ .

Let $C_* \subset C_*^{\text{sm}}(G)$ be the chain complex generated by smooth simplices which are transverse to the barycentric subdivision of f (i.e., f considered as a map $B(\tau) \rightarrow G$). We will use h' to construct a chain homotopy $Q_{f(\tau)} : C_*(G) \rightarrow C_{*+1}^{\text{sm}}(G)$ between $P_{f(\tau)}$ and id ; that is, a linear map which satisfies the identity

$$(2) \quad \partial Q_{f(\tau)}(\gamma) + Q_{f(\tau)}(\partial \gamma) = P_{f(\tau)}(\gamma) - \gamma$$

for all $\gamma \in C_*$. In particular, if γ is a cycle, then $\partial Q_{f(\tau)}(\gamma) = P_{f(\tau)}(\gamma) - \gamma$.

We define $Q_{f(\tau)}$ inductively. The base case is to define $Q_{f(\tau)}$ on 0-simplices; if $g \in G$, let $Q_{f(\tau)}(g) = h'(P_{f(\tau)}(g) - g)$; this satisfies (2). In general, if we have defined $Q_{f(\tau)}$ on simplices of dimension at most k so that it satisfies (2) and σ is a $(k+1)$ -simplex, then

$$P_{f(\tau)}(\sigma) - \sigma - Q_{f(\tau)}(\partial \sigma)$$

is a k -cycle. We can define $Q_{f(\tau)}$ on σ by letting

$$Q_{f(\tau)}(\sigma) = h'(P_{f(\tau)}(\sigma) - \sigma - Q_{f(\tau)}(\partial \sigma)).$$

One problem with this definition is that if σ is a simplex with large diameter, then $Q_{f(\tau)}(\sigma)$ may be very large. On the other hand, if α is a simplicial cycle, then each simplex of α has bounded diameter and we can bound $Q_{f(\tau)}$ on such simplices.

Let

$$c = \max\{\text{diam } f(\Delta) \mid \Delta \text{ is a simplex of } \tau\}.$$

If $S \subset X$, define the r -neighborhood of S to be the union

$$N_r(S) := \bigcup_{s \in S} B_r(s),$$

where $B_r(s)$ is the ball of radius r around s . We will show that for all $0 \leq i < \dim G$ there are numbers $M_i, D_i > 0$ such that if σ is a smooth i -simplex transverse to f whose diameter is at most c , then

- We have

$$\text{supp } Q_{f(\tau)}(\sigma) \subset N_{D_i}(\text{supp } \sigma)$$

- We have

$$\int_Z \text{mass } Q_{f(\tau)}(x^{-1} \cdot \sigma) \, dx \leq M_i(\text{mass } \sigma + \text{mass } \partial\sigma).$$

Note that $x^{-1} \cdot \sigma$ is transverse to f (and thus $Q_{f(\tau)}(x^{-1} \cdot \sigma)$ is defined) for all values of x except a measure zero subset of G .

We proceed by induction. First, consider the case that $\dim \sigma = 0$. In this case, σ and $P_{f(\tau)}(\sigma)$ correspond to points in G , separated by distance at most c , and $Q_{f(\tau)}(\sigma)$ is a curve connecting them with mass $Q_{f(\tau)}(\sigma) \leq w(c)$. Thus let $D_0 = c$ and $M_0 = w(c) \text{vol } Z$ for all d .

Assume by induction that the statement is true for $i \leq k$ and consider the case that σ is a k -simplex which is transverse to f . Then $Q_{f(\tau)}(\sigma) = h'(\rho)$, where

$$\rho = P_{f(\tau)}(\sigma) - \sigma - Q_{f(\tau)}(\partial\sigma),$$

so we consider $P_{f(\tau)}(\sigma)$ and $Q_{f(\tau)}(\partial\sigma)$. By the definition of c , we have

$$\text{supp } P_{f(\tau)}(\sigma) \subset N_c(\text{supp } \sigma).$$

Each simplex of $\partial\sigma$ has diameter at most c , so by induction,

$$\text{supp } Q_{f(\tau)}(\partial\sigma) \subset N_{D_{k-1}}(\text{supp } \sigma).$$

The diameter of ρ is thus at most $2D_{k-1} + 3c$, so the diameter of $Q_{f(\tau)}(\sigma) = h'(\rho)$ is at most $4D_{k-1} + 6c$, and if $D_k := 4D_{k-1} + 6c$, then

$$\text{supp } Q_{f(\tau)}(\sigma) \subset N_{D_k}(\text{supp } \sigma).$$

Furthermore, we have

$$\text{mass } Q_{f(\tau)}(\sigma) \leq w(D_k) \text{mass } \rho \leq w(D_k)(\text{mass } P_{f(\tau)}(\sigma) + \text{mass } \sigma + \text{mass } Q_{f(\tau)}(\partial\sigma)).$$

Let $c'(k)$ be a constant such that for any smooth k -chain γ ,

$$\int_Z \text{mass } P_{f(\tau)}(x^{-1} \cdot \gamma) \, dx = \int_Z \text{mass } x^{-1} \cdot P_{x \cdot f(\tau)}(\gamma) \, dx \leq c'(k) \text{mass } \gamma.$$

This exists by Lemma 5. By induction,

(3)

$$\int_Z \text{mass } Q_{f(\tau)}(x^{-1} \cdot \sigma) \, dx \leq \int_Z w(D_k)(\text{mass } P_{f(\tau)}(x^{-1} \cdot \sigma) + \text{mass } \sigma + \text{mass } Q_{f(\tau)}(x^{-1} \cdot \partial\sigma)) \, dx$$

(4)

$$\leq w(D_k)(c'(k) \text{mass } \sigma + \text{vol } Z \text{mass } \sigma + M_{k-1} \text{mass } \partial\sigma).$$

We can thus let

$$M_k := w(D_k)(c'(k) + \text{vol } Z + M_{k-1}).$$

Then the lemma is satisfied for $c_Q = M_k$. \square

By an abuse of notation, we define

$$Q_{x \cdot f(\tau)}(a) := x \cdot Q_{f(\tau)}(x^{-1} \cdot a)$$

whenever $Q_{f(\tau)}(x^{-1} \cdot a)$ is defined. Note that if this is defined, then

$$\partial Q_{x \cdot f(\tau)}(a) = a - P_{x \cdot f(\tau)}(a).$$

4. FILLING CYCLES IN CARNOT GROUPS

In this section, we apply the approximation techniques of the previous section to construct fillings of cycles in Carnot groups. Let G be an n -dimensional Carnot group, let $s_t : G \rightarrow G, t > 0$ be the family of scaling automorphisms, and let Γ be a lattice in G such that $s_{2^i}(\Gamma) \subset \Gamma$ for $i = 1, 2, \dots$. As described in the introduction, we will build a filling by connecting approximations of different scales; to do so, we will need triangulations of the product of G with an interval, which we shall build out of scalings of a single triangulation. If $(\tau, \phi : \tau \rightarrow G)$ is a triangulation, we define $s_t(\tau)$ to be the triangulation $(\tau, s_t \circ \phi)$. If τ is Γ -adapted, then $s_t(\tau)$ is $s_t(\Gamma)$ -adapted.

Recall that if $S \subset G$ is an open subset, then the scaling automorphisms polynomially expand the volume of S . That is, there is an integer κ such that

$$\text{vol } s_t(S) = t^\kappa \text{vol } S.$$

We call κ the *volume growth exponent* of G .

Proposition 9. *Let τ be a Γ -adapted triangulation of G and let τ' be an $s_2(\Gamma)$ -adapted triangulation of $G \times [1, 2]$ which restricts to τ on $G \times \{1\}$ and $s_2(\tau)$ on $G \times \{2\}$. Let f be a $s_2(\Gamma)$ -equivariant piecewise smooth map $f : G \times [1, 2] \cong \tau' \rightarrow G$ which is such that $f(s_2(g), 2) = s_2(f(g, 1))$. Let $c_0 > 0$ and $k(1), \dots, k(\dim G)$ be such that for all $t > 1$ and all $d' \leq \dim G$,*

$$\sum_{\Gamma \cdot \sigma \in \Gamma \backslash B(\tau')^{(d')}} c_\cap \text{mass } s_t(f_\#(\hat{\sigma})) \leq c_0 t^{k(d')},$$

where c_\cap is the constant from Lemma 4. Let κ be the volume growth exponent of G . Then for all $1 \leq d < \dim G$, if $k(d+1) + k(n-d) > \kappa$ and $k(n-d) \leq \kappa$, then

$$\text{FV}_G^d(V) \lesssim V^{\frac{k(d+1)}{\kappa - k(n-d)}}.$$

Proof. It suffices to construct fillings for all simplicial d -cycles. Let $\alpha \in C_d(\tau)$ be a simplicial d -cycle with mass V . Let $I = I(V)$ be a positive integer to be chosen later. We will construct a filling of α by constructing a triangulation τ_0 of $G \times [1, 2^I]$ and a map $f_0 : G \times [1, 2^I] \rightarrow G \times [1, 2^I]$, then considering

$$P_{f_0(\tau_0)}(\alpha \times [1, 2^I]).$$

We will build τ_0 out of scalings of τ' . The conditions on τ' mean that scaled copies of τ' can be glued together. That is, we extend s_t to a map $s_t : G \times [0, \infty) \rightarrow G \times [0, \infty)$ given by $s_t(g, x) = (s_t(g), xt)$ and define $\tau'_i := s_{2^{i-1}}(\tau')$. For each i , τ'_i is a triangulation of $G \times [2^{i-1}, 2^i]$, and τ'_i restricts to $s_{2^{i-1}}(\tau)$ on $G \times \{2^{i-1}\}$ and to

$s_{2^i}(\tau)$ on $G \times \{2^i\}$. In particular, τ'_i and τ'_{i+1} agree on $G \times \{2^i\}$, and we can glue τ'_1, \dots, τ'_I to obtain a triangulation τ_0 of $G \times [1, 2^I]$.

Likewise, the conditions on f let us extend it to all of τ_0 . We define $f_0 : G \times [1, 2^I] \rightarrow G \times [1, 2^I]$ by letting

$$f_0(p) = s_{2^i} \circ f \circ s_{2^{-i}}(p)$$

for all $p \in G \times [2^i, 2^{i+1}]$. Since $s_2 \circ f \circ s_{2^{-1}}(p) = f(p)$ for all $p \in G \times \{2\}$, this is well-defined and continuous.

Now, for $x \in G$, consider the chain

$$\beta_0(x) := P_{x \cdot f_0(\tau_0)}(\alpha \times [1, 2^I]).$$

Let $\bar{f} : \tau \rightarrow G$ be the map $\bar{f}(y) = f(y, 1)$. The boundary of $\beta_0(x)$ is then

$$\partial\beta_0(x) = P_{x \cdot \bar{f}(\tau)}(\alpha) \times \{1\} - P_{x \cdot (s_{2^I} \circ \bar{f})(\tau)}(\alpha) \times \{2^I\}.$$

This boundary has two pieces: an approximation of α in $x \cdot \bar{f}(\tau)$ and an approximation of α in $x \cdot (s_{2^I} \circ \bar{f})(\tau)$. By Lemma 8, the first piece is usually close to α . We will show that if I is chosen appropriately, then the second term is typically zero, and show that a good choice of x will lead to a β_0 with low mass.

First, we claim that the “large-scale” term

$$\gamma(x) := P_{x \cdot s_{2^I}(f(\tau))}(\alpha)$$

of $\partial\beta_0(x)$ usually vanishes. Let Z be a fundamental domain for the right action of Γ on G , and let I be the smallest positive integer such that

$$2c_0 V 2^{Ik(n-d)} < 2^{\kappa I} \text{vol } Z.$$

By Lemma 5,

$$\int_{s_{2^I}(Z)} \|\gamma(x)\|_1 dx \leq c_0 2^{Ik(n-d)} V < \frac{2^{\kappa I} \text{vol } Z}{2} = \frac{\text{vol } s_{2^I}(Z)}{2},$$

so there is a set $Z' \subset s_{2^I}(Z)$ of volume $\text{vol } Z' > (\text{vol } s_{2^I}(Z))/2$ such that if $x \in Z'$, then

$$\|\gamma(x)\|_1 < 1.$$

Since $\gamma(x)$ is an integral chain, this implies that if $x \in Z'$, then $\gamma(x) = 0$. Let $p : G \times [1, 2^I] \rightarrow G$ be the projection to the first coordinate. If we let

$$\beta(x) := Q_{x \cdot f(\tau)}(\alpha) + p_{\#}(\beta_0(x)),$$

then $\partial\beta(x) = \alpha$ when $x \in Z'$.

We claim that an appropriate choice of $x \in Z'$ leads to a $\beta(x)$ with small mass. Let $Z_i := s_{2^i}(Z)$ and let $f_i : G \times [2^{i-1}, 2^i] \rightarrow G$ be the restriction of f_0 to $G \times [2^{i-1}, 2^i]$. Recall that $s_{2^i}(\Gamma) \subset s_{2^j}(\Gamma)$ whenever $i > j$, so if $\lambda : G \rightarrow \mathbb{R}$ is $s_{2^j}(\Gamma)$ -invariant, then

$$\int_{Z_i} \lambda(x) dx = [s_{2^i}(\Gamma) : s_{2^j}(\Gamma)] \int_{Z_j} \lambda(x) dx = 2^{\kappa(i-j)} \int_{Z_j} \lambda(x) dx.$$

Consider

$$\int_{Z_I} \text{mass } \beta(x) dx.$$

We can break $\beta(x)$ into a sum

$$\beta(x) = Q_{x \cdot f(\tau)}(\alpha) + \sum_{i=1}^I p_{\#}(P_{x \cdot f_i(\tau'_i)}(\alpha \times [2^{i-1}, 2^i]))$$

and bound the average mass of each term. First,

$$\begin{aligned} \int_{Z_I} \text{mass } Q_{x \cdot f(\tau)}(\alpha) \, dx &= 2^{\kappa I} \int_Z \text{mass } Q_{x \cdot f(\tau)}(\alpha) \, dx \\ &\leq 2^{\kappa I} c'_Q V. \end{aligned}$$

Next, for $1 \leq i \leq I$,

$$\begin{aligned} &\int_{Z_I} \text{mass } p_{\#}(P_{x \cdot f_i(\tau'_i)}(\alpha \times [2^{i-1}, 2^i])) \, dx \\ &= 2^{(I-i)\kappa} \int_{Z_i} \text{mass } p_{\#}(P_{x \cdot f_i(\tau'_i)}(\alpha \times [2^{i-1}, 2^i])) \, dx \\ &\leq 2^{(I-i)\kappa} \int_{Z_i} c_0 2^{(i-1)k(d+1)} \|P_{x \cdot f_i(\tau'_i)}(\alpha \times [2^{i-1}, 2^i])\|_1 \, dx \\ &\leq 2^{(I-i)\kappa} c_0 2^{(i-1)k(d+1)} c_0 2^{(i-1)k(n-d)} V \end{aligned}$$

Thus

$$\text{mass } \beta(x) \leq [2^{\kappa I} c'_Q + \sum_{i=1}^I c_0^2 2^{(I-i)\kappa + (i-1)[k(d+1) + k(n-d)]}] V.$$

Since $k(d+1) + k(n-d) > \kappa$, the sum is dominated by the $i = I$ term, so there is a C such that

$$\int_{Z_I} \text{mass } \beta(x) \, dx \leq C 2^{(k(d+1) + k(n-d))I} V$$

Consequently, there is an $x \in Z'$ such that

$$\text{mass } \beta(x) \leq \frac{2C}{\text{vol } Z_I} 2^{(k(d+1) + k(n-d) - \kappa)I} V.$$

Fix such an x . If V is sufficiently large, then the definition of I implies that

$$\left(\frac{2c_0 V}{\text{vol } Z} \right)^{\frac{1}{\kappa - k(n-d)}} < 2^I \leq 2 \left(\frac{2c_0 V}{\text{vol } Z} \right)^{\frac{1}{\kappa - k(n-d)}},$$

so there is a C' such that

$$\text{mass } \beta(x) \leq C' V^{1 + \frac{k(d+1) + k(n-d) - \kappa}{\kappa - k(n-d)}} = C' V^{\frac{k(d+1)}{\kappa - k(n-d)}},$$

as desired. \square

It remains to find groups for which triangulations and maps with the desired properties exist. We will describe two situations in which they do: higher-order filling invariants for the Heisenberg groups and the top-dimensional Dehn functions of Carnot groups which support lattices.

The Heisenberg groups are closely connected to contact geometry. The distribution of horizontal planes in the Heisenberg group forms a contact structure, so results on the flexibility of isotropic maps (see [Gro86] or [EM02]), imply that H has many horizontal submanifolds of dimension $\leq n$. In particular, one can show (see for instance [You08, Lem. 12]) that

Lemma 10. *If H is the $(2n+1)$ -dimensional Heisenberg group, Γ is the $(2n+1)$ -dimensional integral Heisenberg group, and $s_t : H \rightarrow H, t > 0$ is the family of scaling automorphisms of H , then there is a $s_2(\Gamma)$ -adapted triangulation τ' of $H \times [1, 2]$ and a $s_2(\Gamma)$ -equivariant piecewise smooth map $f : H \times [1, 2] \cong \tau' \rightarrow H$ such that*

- τ' restricts to a triangulation τ on $H \times \{0\}$ and $s_2(\tau)$ on $H \times \{1\}$.
- For all $g \in H$, we have $f(s_2(g), 2) = s_2(f(g, 1))$.
- If σ is a d -simplex of τ' for some $d \leq n$, then $f|_\sigma$ is horizontal.

In Lemma 4 of [You08], we constructed a triangulation τ_0 of $H \times [1, 2]$ and a map $f_0 : \tau_0 \rightarrow H$ which satisfy all the conditions above except for the smoothness condition on f_0 ; we only required that f_0 be Lipschitz. In fact, we constructed f_0 to be smooth on simplices of $B(\tau_0)$; we can thus take $f = f_0$ and $\tau' = B(\tau_0)$.

Consequently, we have

Theorem 11. *If H is the $(2n+1)$ -dimensional Heisenberg group and $d > n$, then*

$$\text{FV}_H^{d+1}(V) \lesssim V^{\frac{d+2}{d+1}}.$$

Proof. Since f is horizontal on all simplices of $B(\tau')$ below dimension n , we can let $k(d) = d$ when $d \leq n$. Otherwise, recall that we can write the Lie algebra \mathfrak{h} of H as

$$\mathfrak{h} = \mathbb{R}^{2n} \otimes \mathbb{R}.$$

The scaling automorphism s_t acts by scaling by t on the \mathbb{R}^{2n} component and by scaling by t^2 on the \mathbb{R} component. In particular, there is a c such that if β is a wedge of d vectors in \mathfrak{h} , then

$$\|s_{t*}(\beta)\| \leq ct^{d+1}\|\beta\|,$$

which is sharp, for instance, when β is a nontrivial wedge of $d-1$ vectors of \mathbb{R}^{2n} and a generator of \mathbb{R} . Thus if σ is a d -dimensional simplex of $B(\tau)$ for $d > n$, then there is a c such that

$$\text{vol } s_t(f(\sigma)) \leq ct^{d+1},$$

and we can let $k(d) = d+1$ when $d > n$. Similarly, the volume growth exponent of H is $\kappa = 2n+2$.

Thus, when $d > n$, we have $k(d+1) + k(2n+1-d) > 2n+2$, so Prop. 9 implies that

$$\text{FV}_H^{d+1}(V) \lesssim V^{\frac{d+2}{d+1}}.$$

□

Burillo [Bur96] proved the corresponding lower bound, so this implies that

$$\text{FV}_H^{d+1}(V) \sim V^{\frac{d+2}{d+1}}.$$

In general, if G is a Carnot group and Γ is a lattice in G , there may not be very many horizontal submanifolds in G , but there is always an abundance of horizontal curves. That is, given two points in G , there is a piecewise smooth horizontal curve connecting them; one can construct such curves either directly or by using the Chow connectivity theorem. One can thus construct a triangulation τ' and a map f which satisfies the hypotheses of Prop. 9 and which takes each edge of τ to a horizontal curve. We can thus take $k(1) = 1$. Furthermore, if κ is the volume growth exponent of G and n is its dimension, we can take $k(n-1) = \kappa - 1$. Then Prop. 9 implies Theorem 2.

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